Conjugacy in S_n

How can you tell based on the cycle decomposition of an element of S_n which elements are in its conjugacy class?

Consider a cycle $\sigma = (a_1 \dots a_m)$, and $\tau \in S_n$.

Then what is $\tau \sigma \tau^{-1}$?

If $k \in \{1, \dots, n\}$, then $\tau \sigma \tau^{-1}(\tau(k)) = \tau(\sigma(k))$. We have 2 cases:

Case 1: $k \neq a_i$ for any i (i.e. k doesn't appear in the cycle σ).
Then $\tau \sigma \tau^{-1}(\tau(k)) = \tau(k)$, so $\tau(k)$ doesn't appear in the cycle decomposition of $\tau \sigma \tau^{-1}$!

Case 2: $k = a_i$, then $\tau \sigma \tau^{-1}(\tau(a_i)) = \tau\sigma(a_i) = \tau(a_{i+1})$

That is, σ sends a_i to $a_{i+1} \iff \tau\sigma\tau^{-1}$ sends $\tau(a_i)$ to $\tau(a_{i+1})$
So $\tau \sigma \tau^{-1} = (\tau(a_1)\ \tau(a_2) \dots \tau(a_m))$.

This leads to the following theorem:

Theorem: If $\sigma \in S_n$ has cycle decomposition

$$\sigma = (a_1 \dots a_m)(a_{m+1} \dots) \dots (\dots a_k)$$

then $\tau \sigma \tau^{-1}$ has cycle decomp. $(\tau(a_1)\tau(a_2) \dots \tau(a_m)) \dots (\dots \tau(a_k))$.

Pf: $\tau\sigma\tau^{-1} = \tau(a_1 \dots a_m)\tau^{-1}\tau(a_{m+1} \dots)\tau^{-1}\tau \cdots \tau^{-1}\tau(\dots a_k)\tau^{-1}$, so the cycle decomposition follows from above discussion.

Note that the cycles are disjoint since τ is a bijection:

$a_i \neq a_j \iff \tau(a_i) \neq \tau(a_j)$. $\square$

Thus, two elements of S_n can only be conjugate if their cycle decompositions have the same # of cycles of each length. In fact the converse holds!

Theorem: $\sigma = (a_1 \dots a_{m_1})(a_{m_1+1} \dots a_{m_2}) \dots (\dots a_{m_k})$ is conjugate to $\sigma' = (b_1 \dots b_{m_1})(b_{m_1+1} \dots b_{m_2}) \dots (\dots b_{m_k})$.

Pf: Let τ be the bijection sending each a_i to b_i and every other element to itself. Then $\tau\sigma\tau^{-1} = \sigma'$! $\square$

Ex: S_4 has 5 conjugacy classes, w/ representatives

$1, (1\,2), (1\,2\,3), (1\,2\,3\,4), (1\,2)(3\,4)$, respectively.

More generally, the # of conjugacy classes in S_n is equal to the number of partitions of n:

Ex: • $2 = 1 + 1 \rightsquigarrow S_2$ has 2 conj. classes.

- $3 = 2+1 = 1+1+1 \rightsquigarrow S_3$ has 3
- $4 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1 \rightsquigarrow S_4$ has 5
- $5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1 \rightsquigarrow S_5$ has 7